To integrate x to power x

There is no elementary function for finding the integral $\int x^x \, dx$ (at least to me). However, here is a challenge for you to find the definite integral $\int_0^1 x^x \, dx$.

(a) Given $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$, express x^x in terms of an infinite series.

(b) Use integration by parts, or otherwise, show that :

$\int x^m (\ln x)^n dx = \frac{1}{m+1} x^{m+1} (\ln x)^n - \frac{n}{m+1} \int x^m (\ln x)^{n-1} dx$, where ln stands for the logarithm base e and we don't write the integrating constant for simplicity.

(c) **(i)** Use l'Hospital's rule to show that $\lim_{x \to 0^+} x^m (\ln x)^n = 0$

(ii) Show that $\int_0^1 x^n (\ln x)^n dx = -\frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1} dx$

And $\int_0^1 x^n (\ln x)^n dx = \frac{(-1)^n n!}{(n+1)^{n+1}}$

(d) Prove that $\int_0^1 x^x \, dx = 1 - \frac{1}{2^2} + \frac{1}{3^3} - \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^k}$.

(e) Prove that $\int_0^1 x^{-x} \, dx = 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \cdots = \sum_{k=1}^{\infty} \frac{1}{k^k}$.